The role of implicit and explicit knowledge in understanding mathematics

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Introduction

My grandfather and I share some things in common. We are both called Zoltan Dienes, for example! In addition, we both have an interest in how people learn. I have been particularly interested in how people can learn about sometimes complex structures when they do not necessarily know that they are learning and they cannot say what it is they have learnt. This sort of learning is called implicit learning (see Berry and Dienes, 1993). What I would like to do in this chapter is discuss the relation between implicit learning and my grandfather's principles of education.

Implicit learning occurs pervasively in our life. A good example is how we learnt the grammar of our native language. We know this complex grammar perfectly and can use it in a fraction of a second of produce and comprehend sentences. Yet most people cannot describe any but the simplest rules of the grammar; even linguists do not have a complete grammar of any natural language! Another example is how we learn the rules of social engagement: We can explicitly describe many aspects of these rules, but in general we know more about getting on with people than we can freely articulate. Or consider the experience of listening to music. We can at once judge the style of the music (Beatles or Bach?) even before we know how we have done it; and we can like a style of music without having to be able to say what structural properties of the music lead us to like it. In Chapter Five, my grandfather gives an example from learning mathematics. After experience of a function determining the order of items in a sequence, children can successfully predict the next item while claiming they are just guessing. In fact, the development of implicit knowledge in people exposed to sequences has been extensively explored by psychologists interested in implicit learning (e.g. Reber, 1993).

Implicit learning is unconscious learning in the sense we are not conscious of the knowledge we have learnt. Explicit learning is learning that occurs when we are conscious of the knowledge we have learnt; we can then in general describe the knowledge that guides our actions. For example, knowledge of the recipe I follow in cooking a cake is often explicit. Following a worked maths example in order to solve a related problem would involve explicit knowledge. In Chapter Five, when people had learnt a particular group structure involving yellow and red cards, and could say "If I play a yellow, you show the same card that you showed before, but when I play a red, you change the colour", these people had demonstrated that they had acquired explicit knowledge.

Learning about a domain will often involve both implicit and explicit components. Given that one important aim of my grandfather's methods is to produce understanding, it is important to consider what role implicit and explicit learning may play in developing an understanding of a concept.

What is it to understand a concept?

What is it to understand something? This is a surprisingly difficult question. But we can say some things. We can say that understanding, our grasp of something, admits of degrees. We can understand a concept more or less well. The extent of our understanding of a concept is directly related to how systematically we can use the concept. A person's use of a concept is systematic if they can use the concept in all the contexts to which it applies. If you tell me the definition of a mathematical group, I can understand the conditions which you specify, and that alone gives me a degree of grasp of the concept. If I can recognise the group structure of various concrete situations then I demonstrate a deeper understanding. Similarly, I can demonstrate my understanding of a cardinal number, e.g. the number six, by counting a set of objects and telling you, in answer to the question "How many are there?" that there are six. If however, after you have spread the same objects out on the table I believe there are more than six objects now (as Piaget showed young children do), then my failure to grasp there are still six objects there reveals a lack of full understanding of what it is for there to be six objects. I am not able to apply the concept in all the contexts to which it is applicable.

Intellectual discovery often consists in seeing old patterns in surprising incarnations; surprising because we did not know they were there before. That is, the creative act of intellectual discovery often consists in the identification of a well known pattern, or mathematical concept, in a new guise. The person who makes the discovery demonstrates their deep understanding of the concept; creativity and understanding go together. A person has a deep understanding of a concept if they can apply it in novel contexts. If a person does not recognise the pattern in situations where the pattern does in fact manifest, the person's grasp of the concept is not so great. If I think that a quadratic equation is just something I know how to deal with when the teacher sets me certain types of problems to solve, and I just use the standard formula for solution then I do have some hold on what a quadratic is. But because I will fail to identify this pattern in many situations where it may be - hidden in plain sight - my grasp of the concept is limited.

These comments apply not just to mathematics. To quote from the philosopher Millikan (1984), who was not explicitly talking about mathematics: "My hold on what I am thinking of depends upon how versatile and how reliable I am in performing acts of correct identification of the referents of my thought .. . . having the capacity to identify what I am thinking of via percepts as well as via thoughts and language is another way to increase my hold on what I am thinking of. . . if I can recognise you on the street, or recognise your voice on the phone as well as knowing your name and something about you, I have better hold on who I am thinking of when I am thinking of you than if I cannot (p. 252)." According to Millikan (1998), having a concept means being able to identify it, more or less reliably, in its different instantiations. Millikan proposed her account as a theory of having concepts of substances; I have found Millikan's approach very useful in trying to understand mathematical understanding as well.

Understanding and implicit and explicit knowledge

Understanding mathematics cannot just be the ability to manipulate symbols according to mechanical rules. Searle (1980) employs his "Chinese room argument" to caricature the notion that understanding could be produced by following explicit rules. Searle considered the case of understanding stories. Take an English person who understands no Chinese. Lock him in a room where there is a huge rule book written in English. He is given a piece of paper with a story in Chinese on and then a series of questions about the story, all in Chinese. He has no idea what any of the Chinese characters mean, but the rule book tells him how to manipulate the symbols so as to produce an answer to each question; in fact, an answer that a Chinese person would recognise as the correct answer to the question. The person can follow all the rules perfectly, and thereby produce correct answers, but he has no idea what any character means, or what any of his answers mean. So understanding is not just following rules.

Understanding is lacking for the person in the Chinese Room because the person could not identify the referent of any of the characters by any perceptual (visual, tactile, etc) means, even if the person were actively manipulating the referent in question. For example, the person could be following rules appropriately for the character for "horse" in answer to questions written in Chinese; but the person would not know that the character could be used to name a horse the person was looking at, riding on, etc. Further, the person could not identify the real world patterns referred to by combinations of characters (phrases, sentences, e.g. "this is a horse") by any perceptual means; when looking at a horse they could not identify this situation as a referent of the sentence "this is a horse".

Grasping what a word means does not just entail being able to give an explicit definition. I show my understanding of the definite article ("the") and indefinite articles (e.g. "a") by using them appropriately in the contexts to which they apply; I identify each of these individual contexts as appropriate for using the words whenever I use the word. It is not at all necessary that I be able to give an explicit (dictionary) definition to demonstrate my understanding. In fact, if a person whose native language does not have articles (e.g. Chinese) comes later in life to learn English, they learn the dictionary definition and often can recite the definition better than most English people. Nonetheless, their USE of articles is often inappropriate; the dictionary rules just don't cover the subtleties of actual language use. That is, often our grasp of how to identify appropriate situations for using words is entirely implicit.

If all I could do was give a dictionary definition for every word I used, then I would be like the person in the Chinese Room: I would know how to relate a lot of symbols together, but I would have no understanding. Something must take the free-floating symbols and connect them to the world; that is, to "ground" them in the real world. There must be a mechanism that goes from our perceptual interaction with the world to the symbols (Harnad, 1990).

In general our understanding of the symbol is not exhausted by our explicit knowledge of the conditions of using the symbol (since such explicit knowledge just consists in uttering more symbols); that is, our understanding must be more than explicit definitions and descriptions. Therefore, the connection between perception and symbol consists of implicit knowledge. Here are some examples of how our perceptual mechanisms yield categories and labels in ways we cannot describe. By what features do you recognise particular people's faces? Consider what happens when someone has a haircut, or changes their glasses, or makes some other minor change; often we cannot say what has changed, at least for some moments. Nonetheless, what it is that has been changed has been registered somewhere inside us in order for us to know that there has been change. What makes a face beautiful? Even plastic surgeons have had a hard time discerning the relevant facial properties; it seems in fact that perception of beauty has a lot to do with the implicit perception of the so-called Golden Ratio, a particular mathematical relationship (Bates & Cleese, 2001). Much of the way we categorise the world is based on implicit knowledge.

Some ways by which we identify the referents of words, or identify the appropriate conditions for using a word, are implicit. But some ways are also explicit. Sometimes we do appropriately use words and concepts by use of knowledge we can explicitly state. You might ask why did I call Tom a bachelor? I might answer that is because Tom is an unmarried adult male, thereby giving an explicit account of my use of the word bachelor.

Understanding a concept is for many concepts shown by being able to identify the referent of the concept in any of the contexts to which it is applicable. There must be methods of identification that involve implicit knowledge in order to connect with the real world; but other methods of identification can be explicit (Millikan, 1984). Sometimes we require both implicit and explicit identification methods. I might show my understanding of how to ride a bike by riding a bike. But I could still claim "I do not understand how I ride this bike" because I cannot say what I do to control it and stay upright, other than I just do it. So sometimes understanding involves making explicit our implicit knowledge. Likewise, when asked by a second-language student of English, why do I use a definite article there, I might after reflection consider that I do not really understand how definite articles are used - because I could not make explicit my implicit knowledge.

A person has a good grasp of some structure in mathematics if they can identify it across its different possible perceptual embodiments (and hence use that structure in at least many of the contexts in which that structure is relevant). When a person has a good understanding, such identification would in general consist of both implicit and explicit processes.

Understanding could initially consist of simply learning implicit and explicit procedures for dealing with the structure in many different contexts. Developing such skills can be valuable per se. But for my grandfather's methods, specific ways of dealing with particular embodiments are merely the stepping stones to developing an understanding of the concept per se. In fact, initially, these contextually-embedded processes do not identify the same structure as being the same in the different embodiments. The person simply acquires practical knowledge - implicit or explicit - of dealing with the structure-in-context. For example, the person may in fact be dealing with a structure isomorphic to the symmetries of the cube, but the person only sees it as dealing with a situation involving, say, a father, daughter and son, and various interactions between them. The person at this point does not see the relation between this structure and the cube. Once the person has built up contextually-embedded knowledge about the structure in different embodiments, the person can be asked to map between the embodiments. When the person has successfully noticed the isomorphism between the embodiments, the perception of the commonality of the embodiments is the person's first grasp of the abstract mathematical structure itself. (This is what Millikan, 1998, also requires for us to have a concept: We must be able to identify it as the same thing, as we track it across different contexts.) When the person has noted the mapping between the embodiments, their knowledge of how to deal with each of the different contexts becomes identification procedures for the structure in question: Now the particular embodiments are seen AS embodiments of the same structure. At this step, the person has acquired an understanding of the structure, an ability to see it in different contexts, viz those the person has just been exposed to. This last step, of establishing the isomorphism, is very important, but sometimes missed out in student-centred contextual learning in practice.

Implicit and explicit knowledge in learning particular embodiments

Now I want to turn to some details of the role implicit and explicit knowledge plays as a person learns a mathematical structure in a particular concrete embodiment. Consider the person learning the symmetries of the cube but as embodied in a game about a father, a daughter, and a son, and their interactions. In one sense of implicit, the structure of the symmetries of the cube is implicit in the knowledge the person has about the father, daughter, and son; but this is a different sense of implicit than the one used in the implicit learning literature. The person does not have knowledge which is unconscious about the relation between the cube and the domain of the father, daughter and son; the person does not have any knowledge, conscious or unconscious, so no implicit knowledge of the cube is involved. The person just has knowledge about the structure of the family. The knowledge might be explicit (the person can say why certain operations yield certain results) or implicit (for example, the person might believe they are just guessing); but whatever its form, it is about the father, daughter, and son. Is there an optimal mix of implicit and explicit learning in learning about a particular embodiment?

Perhaps surprisingly, the development of implicit and explicit knowledge in children has only just started to be investigated by experimental psychologists. But it appears that in some cases, knowledge is originally learnt implicitly and only later is made explicit in the normal course of development (e.g. Karmiloff-smith, 1992; Clements & Perner, 1994; Ruffman et al, 2001). For example, Clements and Perner and Ruffman et al looked at how children understand that other people can have false beliefs. According to a story the children heard, a mouse falsely believes where some cheese is; when the mouse next looks, will he look where the child knows the cheese actually is, or where the mouse last saw the cheese placed? Children before about three years old state that the mouse will look where the cheese really is; older children appreciate the mouse can have a false belief, and state the mouse will look where the mouse last saw it placed. There is a period of a few months before they fully understand false belief, when children will state the mouse will look where the cheese actually is (that is, their verbal response fails to indicate the children understand false belief), but the children themselves spontaneously look to where the mouse should falsely believe the cheese is (i.e. the location children should indicate if they understand that others can have false beliefs). The looking was taken as an indication of developing implicit knowledge of false belief, implicit knowledge that presaged the later explicit knowledge. Further, Ruffman and his colleagues have shown that it is this initial implicit stage in understanding false belief that is missing in autistic children. These children can come to explicitly understand about other people's beliefs and false beliefs, but it is a effortful hard process; most of us can just see intuitively how people can come to have true and false beliefs. Karmiloff-smith developed a general theory of learning in children; she argued that a typical sequence in many domains was for the child to first acquire implicit knowledge, and then, after the child has mastered the task behaviourally, the knowledge gradually becomes explicit.

Sometimes the development of knowledge goes in the opposite direction, from explicit and conscious to unconscious, or at least from consciously controlled to automatic. For example, when we learn to tie our shoe laces, we may laboriously work out each movement; but after many repetitions practicing the task we can perform it without thinking. Sometimes explicit knowledge may serve as a scaffold to help relevant implicit knowledge to develop, as when learning the grammar of a second language explicitly. It is an open question of what the optimal strategy is educationally in terms of the mix of implicit and explicit knowledge encouraged at different points in time (and with what individual differences?). This is a question that experimental psychologists interested in implicit learning are only just beginning to investigate.

Mathews and colleagues (1988) tried to teach a type of rule structure to adults either in an implicit way, an explicit way, or a mixture of both. People were presented with strings of letters, like the following example:

XCSS.TPVV

There were always an ordered set of four letters, a period, then another ordered set of four letters. Any of the letters X, T, P, C, S and V could occur in any order in the first set. The letters in the second set were fixed by this choice. The letter in the first position of the first set determined the letter in the first position in the second set, the second position determined the second position, and so on. The rules were X goes with T, C with P, and S with V.

Mathews encouraged an implicit form of learning by not telling people there were rules and just asking them to try to remember particular strings for short periods of time. He encouraged an explicit form of learning by asking subjects to try to work out the rules and indicate the parts of a string where rules may be violated. Some subjects just learnt under conditions encouraging implicit learning; others under conditions encouraging explicit learning; and others learnt for half the time under implicit conditions followed by half the time under explicit conditions. It was the latter subjects, encouraged to learn first implicitly and then explicitly, that showed the greatest subsequent ability to classify new strings. Mathews points out that it is not known how general this synergistic effect of implicit and explicit learning is, but it is consistent with the claim that "forming a hypothesis too early may prevent best use of the information available to a person" (Mathews et al, 1988, p. 1098). Sometimes it is indeed useful to simply be exposed to stimuli and implicitly absorb the structure before trying to be analytical, even in adults.

The above results at least urge caution in the practice of making children approach tasks analytically as a matter of routine. Getting an intuitive feel for materials, or the way a game works, may be a vital preliminary stage for at least some children in at least some domains.

Knowledge is fully explicit when the person represents the knowledge as knowledge that they have (see Dienes & Perner, 1999). In this case, the person can say with confidence e.g. that the next element in the sequence is a red triangle, or (at the next level) that the same structure applies in the different embodiments. If the person does not possess fully explicit knowledge, they may just believe they are guessing (even though they consistently answer questions correctly).

Work with adults learning artificial grammars indicates that often subjects come to be able to answer consistently and correctly while believing they have no knowledge whatsoever, they are simply guessing. With more experience on the task, their confidence comes to match their performance more and more closely, even when they are not given feedback by an external person as to whether they are being correct or not. When learning artificial grammars, people do not need verbal exhortations to be more or less confident in order to bring confidence in line with performance; simple experience is enough. Implicit knowledge will gradually be made explicit with experience (though often there remains a small or large residue of purely implicit knowledge).

Once a person has acquired various identification procedures for the concept-in-context, by my grandfather's methods, they are then encouraged to attempt an appropriate mapping between the domains (to establish they can track the same structure across the contexts). To what extent can this happen implicitly or explicitly?

Implicit and explicit knowledge in forming an abstract concept

Often, when one really understands a concept one can generate novel explanations, novel applications, and identifications in novel contexts. This can happen more or less explicitly. In one way of achieving this, the person could simply determine analogies with existing embodiments that are known about. The mental procedures that determine these analogies constitute the person's understanding of the abstract structure itself. The knowledge is implicit in one sense: The abstract structure is represented implicitly in the procedures used in identifying the structure in different domains by the process of analogy. When using analogy, the person may be able to say that there is a common structure, without being able to say what the structure is, except by example.

Alternatively, the person could form an internal mental model, or representation, of the structure, where elements and relations are represented by variables that can be bound by different perceptual embodiments, and the model's function is to represent the particular structure abstractly. This constitutes an explicit representation of the structure itself. .

So the abstract structure can be represented implicitly or explicitly as an abstract structure. In either case, the knowledge can be implicit or explicit as knowledge. For example, knowledge by analogy would be fully implicit as knowledge if the person did not know that analogy was the process being used and they believed they were just guessing. Alternatively, the person may be quite aware of using analogy and know that they have this knowledge.

Real mathematicians of course have explicit knowledge of explicit abstract structures. But real mathematical thinking rarely proceeds by strict formal manipulation of rules and symbols. Koedinger and Anderson (1990) found that expert mathematicians, even in planning something so outwardly formal as a mathematical proof, engaged in intuitive leaps of inference enabled by concrete, perceptually-based knowledge representations. Fully analytic knowledge of abstract structures is supported by perceptual knowledge of particular embodiments, even in the expert – perhaps, especially in the expert. Consistently, Heffernan & Koedinger (1997) propose what they call the inductive support hypothesis: "Formal knowledge grows out of prior inductive experience which provides a semantic foundation in the form of back-up strategies for retrieving, checking, or rederiving formal knowledge."

Thus, knowledge of different embodiments of a mathematical structure actually enables a person to find out more about the mathematical structure. Each embodiment provides it's own type of inspiration and suggestions about properties of the structure. Explicit knowledge of the embodiments helps in consciously thinking about the structure; implicit knowledge of the embodiments may play a role in intuitive leaps that seem to come from nowhere (Davies, 1992).

Conclusion

I have tried to provide a framework for seeing how the distinction between implicit and explicit knowledge relates to learning mathematics, especially in light of my grandfather's methods. I have found it helpful to view having the concept of a certain mathematical structure to be the ability to track that structure across different embodiments of it (cf Millikan, 1998). This naturally leads to a method of education: Present the structure in different embodiments and also make sure the person identifies it as the same structure across the embodiments (and, my grandfather specifies: in that order, at least at a first pass at grasping the concept). A person can have means of identification that are implicit or explicit. Ultimately, there must be both forms of knowledge: The explicit to provide the analytical understanding, but also the implicit to ground the knowledge in the real world and give it meaning.

I am myself no expert in children's understanding of mathematics in particular (I am an expert in implicit learning generally), but I am glad to see experts' views that fit this approach. Nunes & Bryant (1996) provide an excellent overview of how children's understanding of mathematics develops in definite ways that allow them eventually to track the same mathematical structures across different situations. Koedinger and others from Anderson's research group at Carnegie Mellon have explicitly addressed the role of implicit knowledge and processes and perceptually-embedded knowledge in mathematical thinking and education. From a practical point of view, we have made great strides, but finding the right mix of encouraging implicit and explicit knowledge for different domains for different individuals is still in good measure up to the implicit knowledge of teachers.

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