Synthesizing the legacy of Varga and Dienes

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When Varga met Dienes…

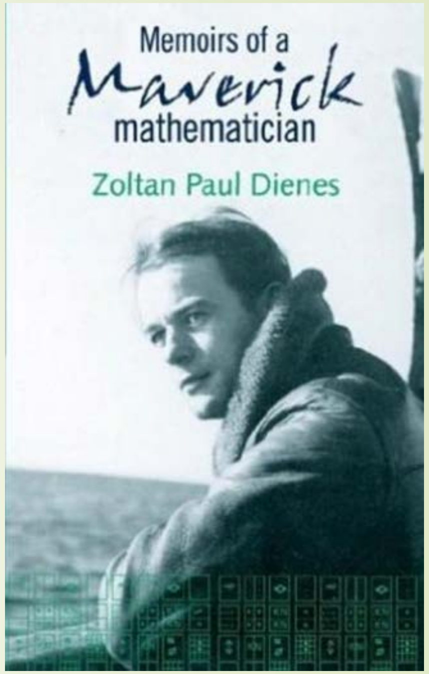
"During the nineteen-fifties and early sixties I was charged at the Budapest University with courses on mathematics education to prospective teachers of grade 5 through 12.

I felt that my words needed **factual support**; this is why I decided to **test my suggestions** with an **average group** of pupils from **grade five** in five weekly hours." (Varga, 1988)

"After three years of our intensive work, Z. P. **Dienes arrived on the scene** during the summer of 1960. He came from Cracow where he had participated at a meeting of the CIEAEM. In Budapest he delivered a lecture at the Second Hungarian Mathematical Congress (where I, too, reported on my experiences), and **conducted some demonstration lessons**. What **he told and showed** us convinced me of **the necessity of a new start**, one **with younger children** and a completely **different organization**."

Z. P. Dienes, the "Maverick" (page 317)

"The mathematics I was bringing into the schools was really a Trojan horse. It was **not just mathematics**, it was a **way to look at what learning is all about**, or even more fundamentally, what knowledge is all about."

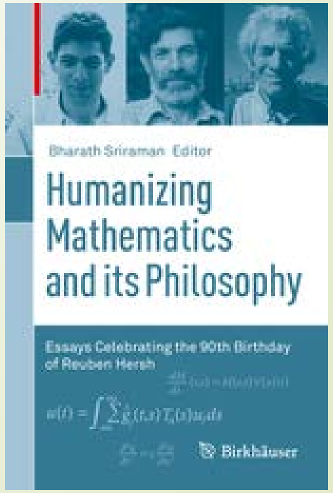


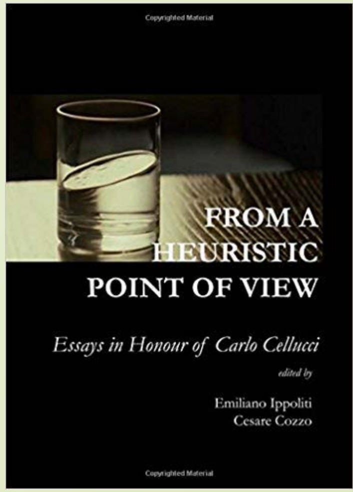
- " To 'know' something surely is to know ***how to handle it***. Handling means action: *present action or at least past action*, remembered accurately, burnt into our person as internalized action. So if knowledge is internalized action, then ***learning*** *must be the* ***process of internalizing such action***."
- "If there is no action, then there is nothing to internalize, so no learning of any permanent nature can happen. It is ***philosophies such as these that climb out of the Trojan horse*** once it is smuggled into the educational system under the guise of essential learning, such as the learning of mathematics."

Maverick philosophy of Mathematics

Representatives of the "Maverick Tradition" (Kitcher and Aspray, Reuben Hersh, Carlo Cellucci) maintain that

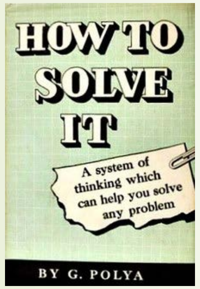
- "mathematics is a **human activity**"
- "intelligible only in a **social context**"
- "mathematical objects exist only in the **shared consciousness** of human beings"
- "are concerned with 'the philosophy of **mathematical practice**' [...]
- "mathematical practice includes **studying**, **teaching** and applying mathematics" (Hersh 2014, 59)

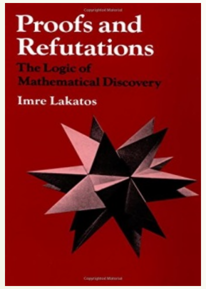


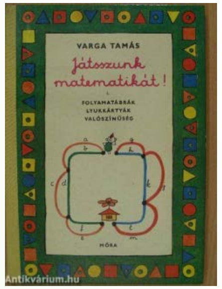


Roots of Mathematical Practice

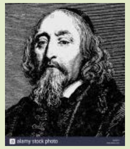
"I [planned to write] so that the learner may always see the inner ground of the things he learns, even so that the source of invention may appear, and therefore in such a way that the learner may understand everything as if he had invented it by himself." (G. W. von Leibnitz: Mathematische Schriften, ed. by Gehardt, vol. VII, p. 9.) quoted by Polya





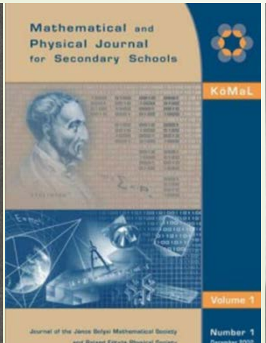


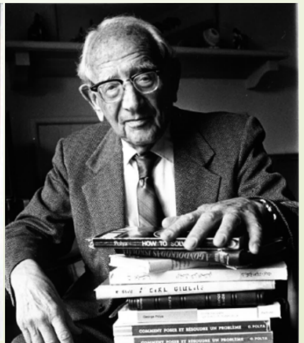
"To make people enjoy learning everything."

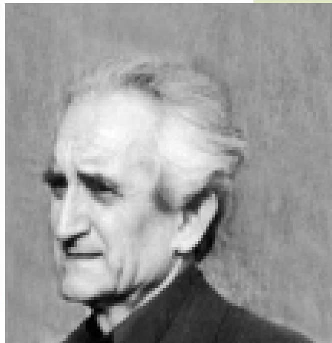


"To transform all treadmills, so called schools, ..., into a playground."(J. A. Comenius: Pampeadia, 1666)

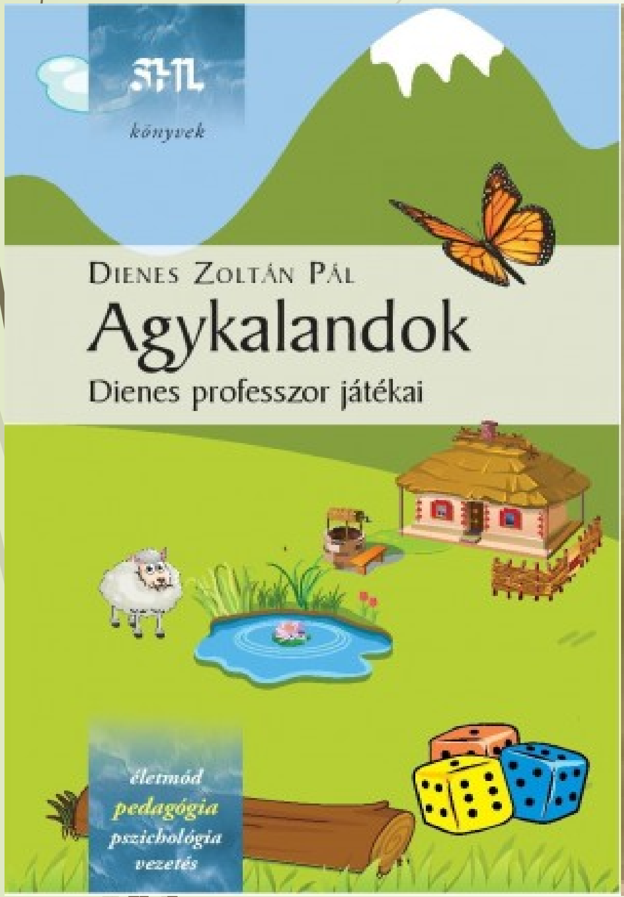
The 'Hungarian Tradition'





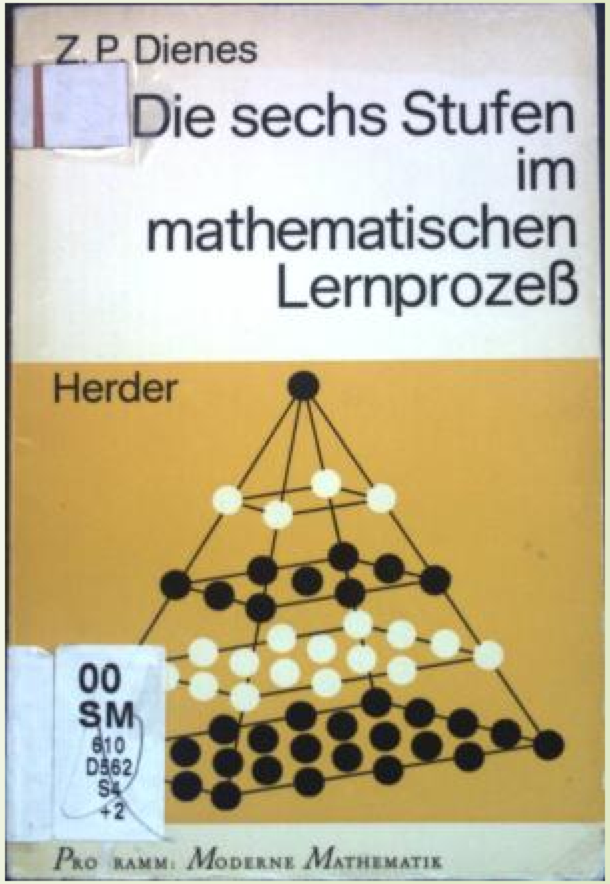


"Give me a mathematical structure and I will turn it into a mathematical game!" (Z. P. D.)



Dienes's six stages of learning

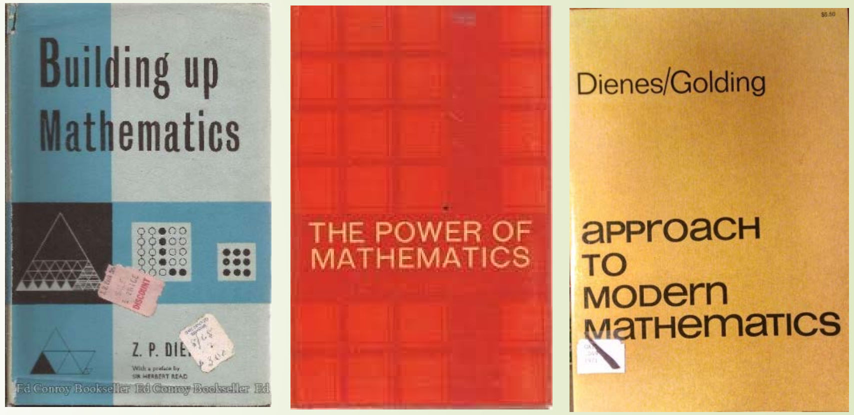
- Stage 1) ***Free Play***
- Stage 2) ***Rule-based Games***
- Stage 3) ***Comparartive Structuring***
- Stage 4) ***Representation***
- Stage 5) ***Symbolization***
- Stage 6) ***Formalization***



Dienes's Principles

Dienes's four main principles (1960, 1964, 1971):

- the **Constructivity Principle**
- the **Dynamic Principle**
- the **Perceptual Variability Principle** (or **Multiple Embodiment Principle**)



- the **Contrast**, or **Mathematical Variability Principle**
- They are complemented by the *Function Principle*, the *Interdisciplinary Principle*, and the principles drawn from the nature of mathematics, i.e., *Abstraction*, *Generalization*, and the ***Deep-end Principle***.

Work with manipulatives

Dienes's Multibase Arithmetic Blocks

The **base ten arithmetic blocks** shown on the picture are used *typically* **instead of** Dienes's **Multibase** Arithmetics Blocks.

Variation of the base (and other factors) demonstrates **Dienes' principle of multiple** embodiment.(Dienes, 1964, p. 40)

Work with manipulatives

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Dienes's **Multibase** Arithmetics Blocks.

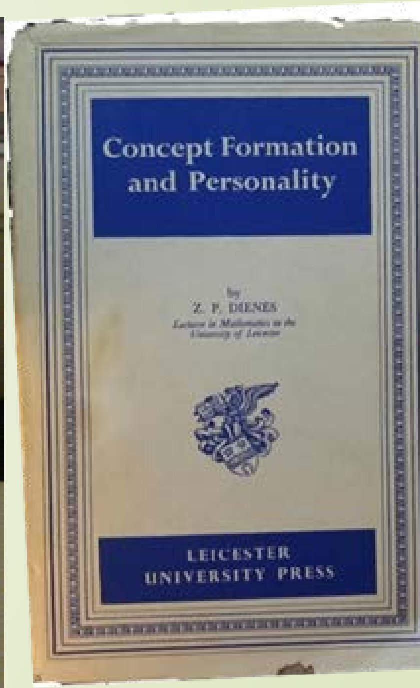
Variation of the base (and other factors) demonstrates **Dienes'** principle of **multiple** embodiment.(Dienes, 1964, p. 40)

Varga on teacher training:

“Imparting new knowledge is relatively easy. Difficulties arise when trainees are supposed to **unlearn** *obsolete concepts, to abandon familiar views, to* ***change habitual practices***, or - most difficult though most important of all - to **change their attitudes**. I mean, for instance, accepting children as *fellow-learners* whose ways of thinking, silly as they seem, merit serious attention - not a standard attitude on the part of Hungarian teachers, I must say.“ (Varga, 1988)

How to change attitudes in teacher education?

Can we teach prospective teachers **in a similar way** as they should theach students based on Dienes’s and Varga’s principles?



Many teachers rely on their *habitual practices* recalling divisibility rules (in base 10).

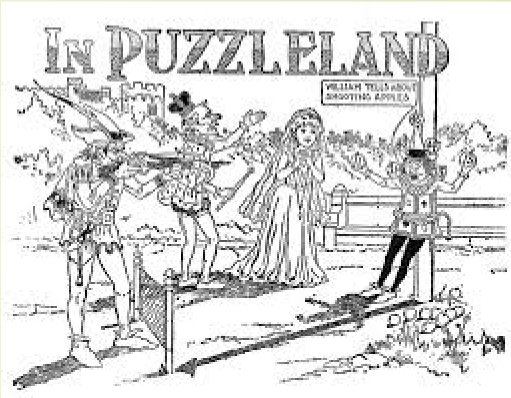
- Can these rules be explained?
- Can new “rules” be constructed?
- Can teachers be challenged in an environment that is **unfamiliar** or **puzzling** to them so that they can not rely on their memory? Can they be turned into “fellow learners”, exploring new mathematics together?
- **Do manipulatives help** adults as much as pupils in mathematical **investigations**?
- If they do, how we can use them?

Outline of a lesson on divisibility and division for teachers using Varga's following ideas:

"One of the most important means of new-style mathematics teaching is to **puzzle children**, indeed **to confuse them**, again and again. […]

Those who are often '**confused**' […] are **more likely to think independently** and less likely to accept an idea or a statement on mere authority.[…]

Their struggle with words helps their thoughts to ripen; **the need for successful communication** improves their ways of expression." (Servais & Varga, 1971)



Dienes's Puzzle Land

Inducing Confusion

Representing and testing the (counter)examples with ***Dienes multibase arithmetic blocks***

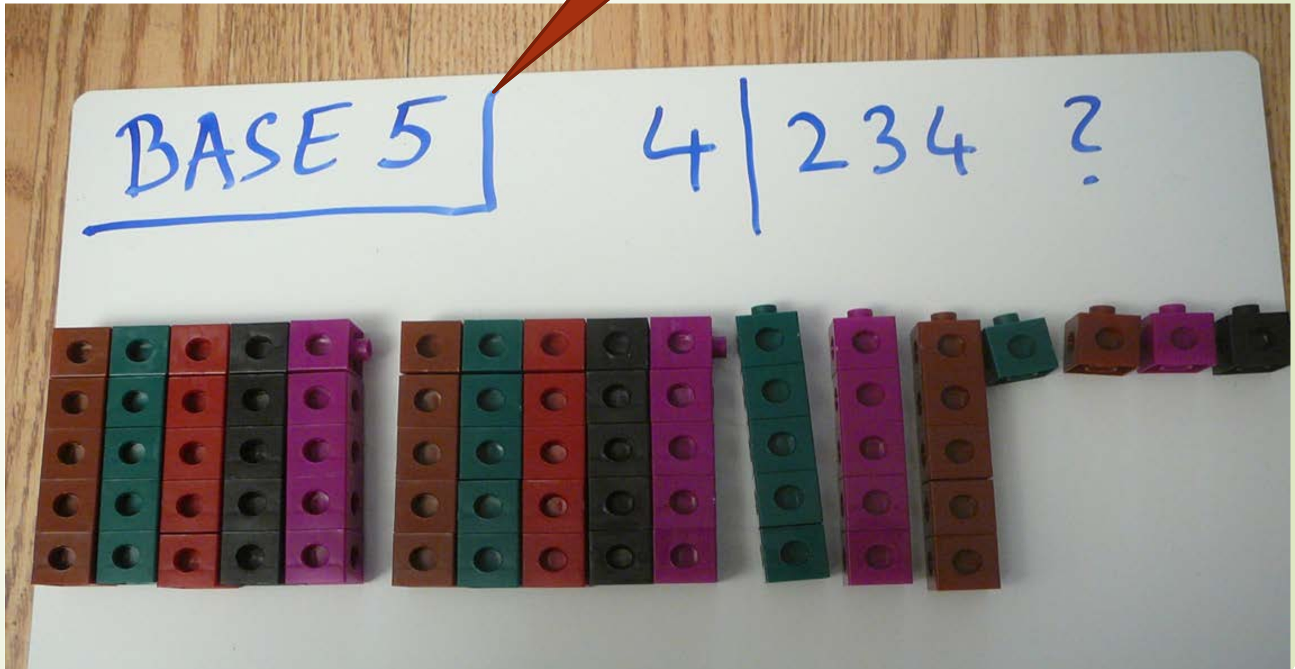
Polling the Classroom:

Which propositions are true in base 5?

- 4 | 444
- 4 | 234
- 4 | 121
- 4 | 144

Multiple experience

Constructivity Principle



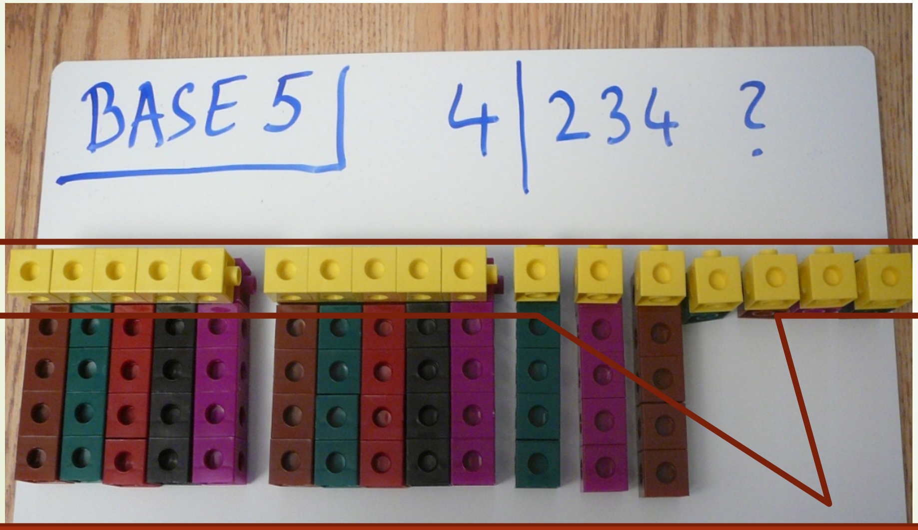
Rule-based search for generalizability

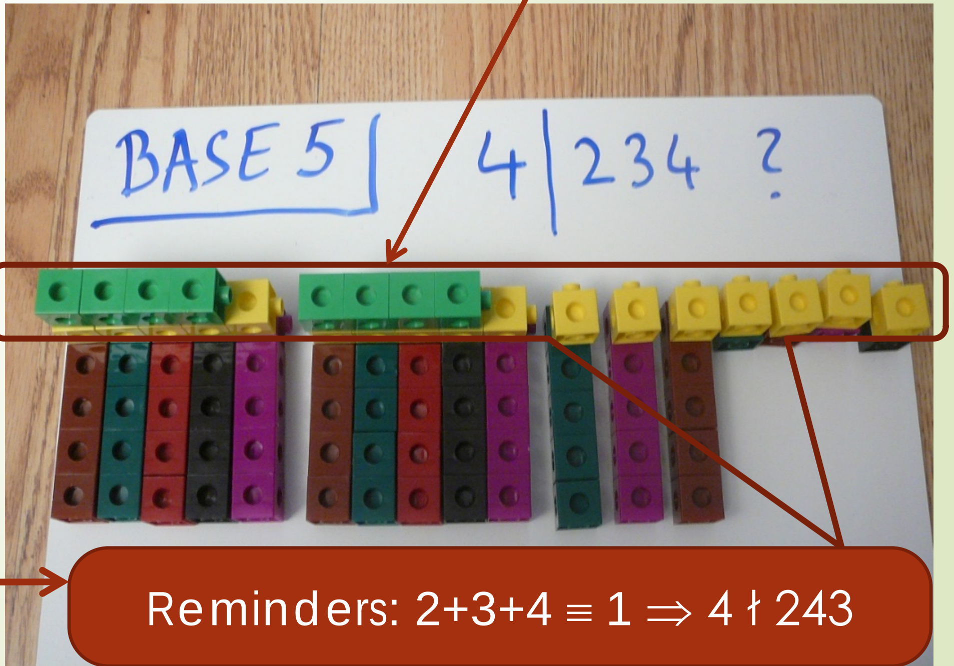
Whole-class discussion of questions:

How do you answer the question based on *Dienes multibase arithmetic blocks*?

- Analogy to base ten “rules” you know?
- Generalization?

Rule-based ***Comparative Structuring***





Reminders: $(4+1)+(4+1)+3+4 = 4+4+4+4+1$

Reminders: $2+3+4 \equiv 1 \Rightarrow 4 \nmid 243$

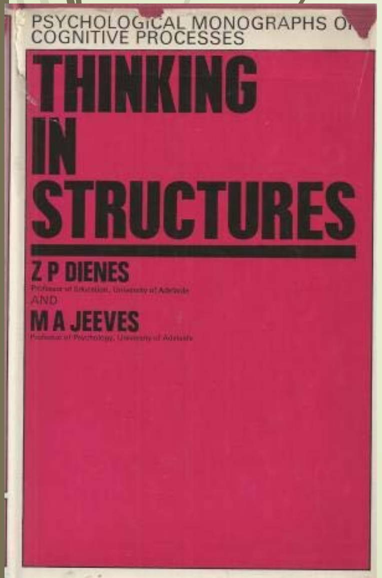
The Deep-end Principle

Classroom Teaching

Teacher Training

- The policy of **teaching a structure by also teaching at the same time a superordinate structure including the one to be taught**.
- A method for introducing concepts, *in case of observed analytic tendencies*
- Builds on the difference of linguistic and arithmetic concept formation
- Grasp the general concept first and concretize it later

- »*The use of the different bases is an application of the "**throwing them in at the deep end**" principle ...validated as a result of a number of experiments in different parts of the world.* « (Dienes, 1966 UNESCO p. 84)
- *"The psychological justification of this is that structures are very much more easily learned if they are embedded in other structures"* (*Ibid*. p. 27)



Applying the Multiple Embodiment Principle

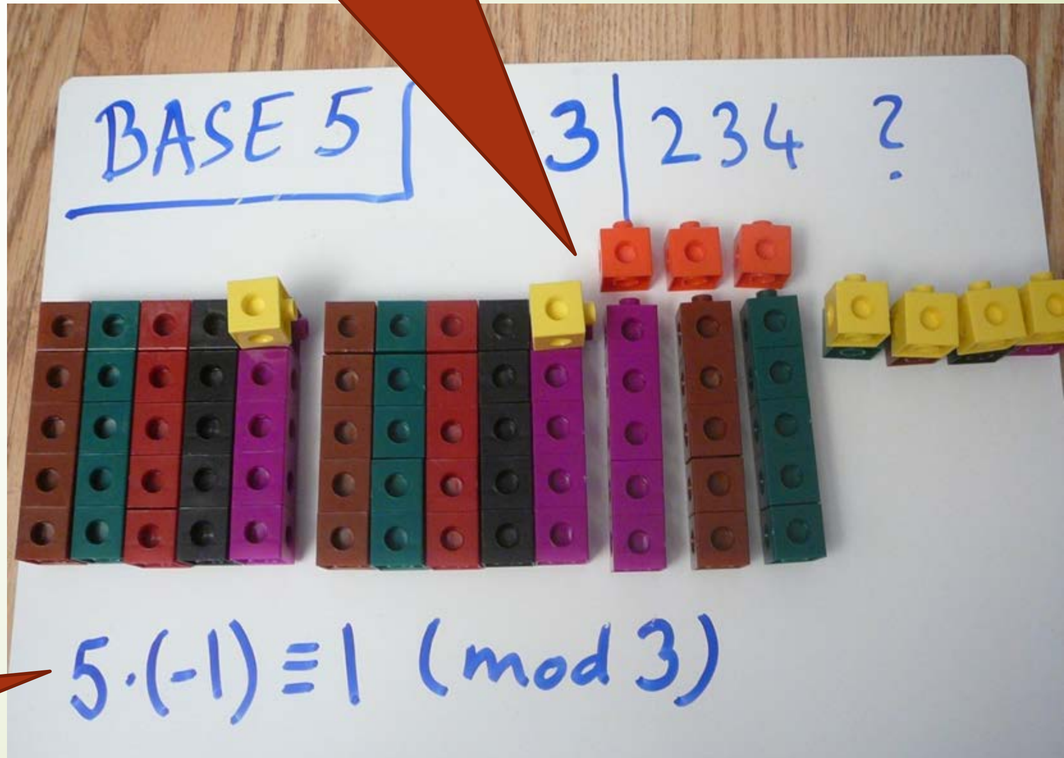
- Varying the divisor
- New challenge to work on in groups: What is the “divisibility rule” for division by 3 in base 5?
- Constructing the “rule” using the blocks
  - Emphasis on developing clear language for communication of the extracted rules
  - introducing symbols for the abstract components
  - checking the results of abstract rules
- deducting the rules in terms of symbolized properties

Perceptual Variability Principle

Representation
(Color-code)

Symbolization
(including verbalization)

Formalization



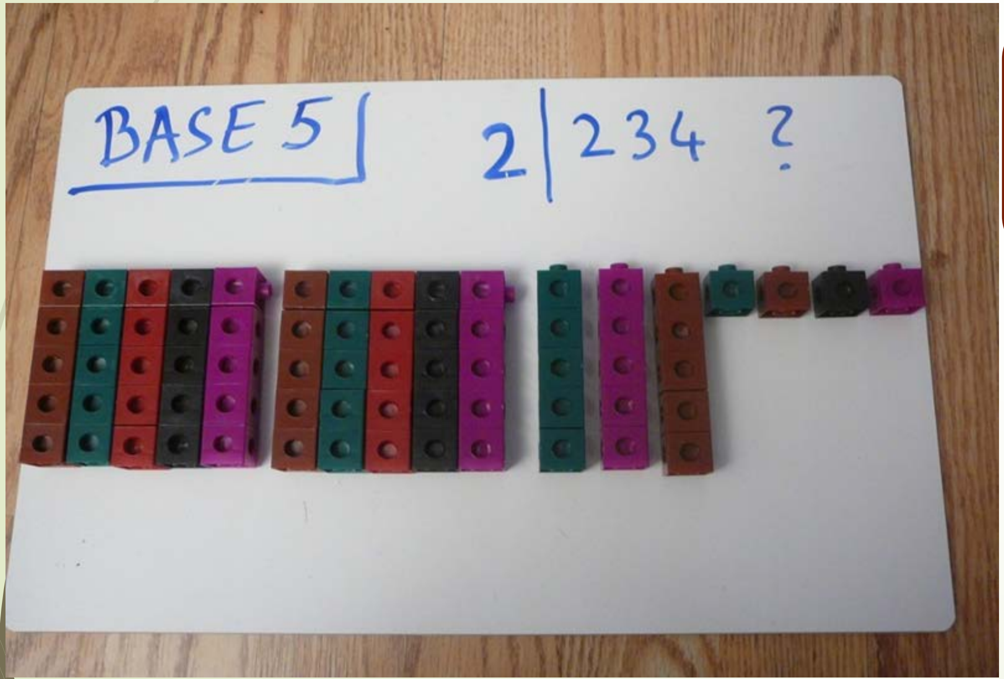
Applying the Variability Principles

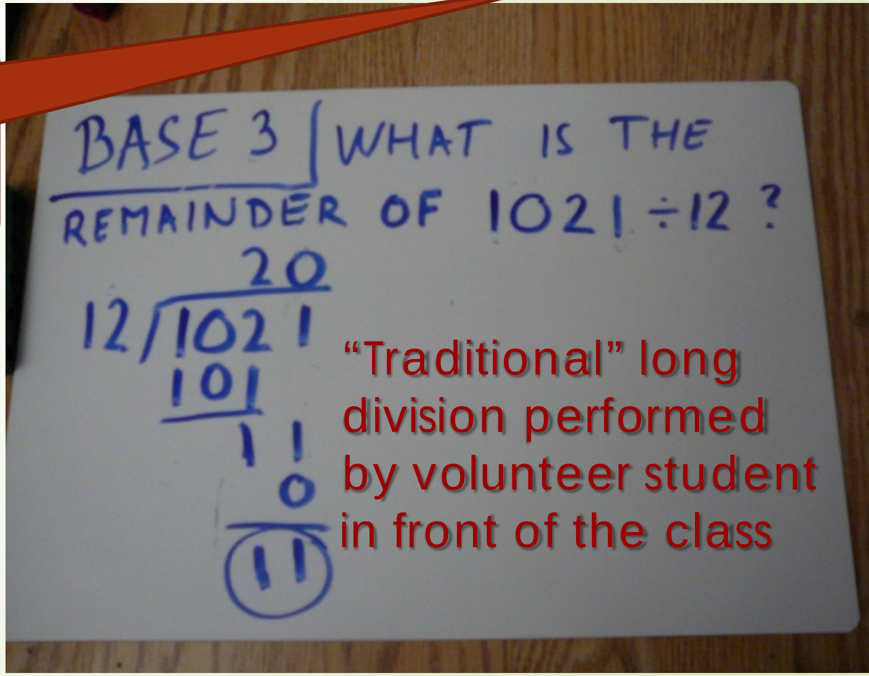
- What is the "divisibility rule" for division by 2 in base 5?
- Varying the divisor

Perceptual Variability Principle (again)

- Predict the reminder!
- Varying the base
- Varying the dividend

Contrast Principle (Mathematical Variability Principle)

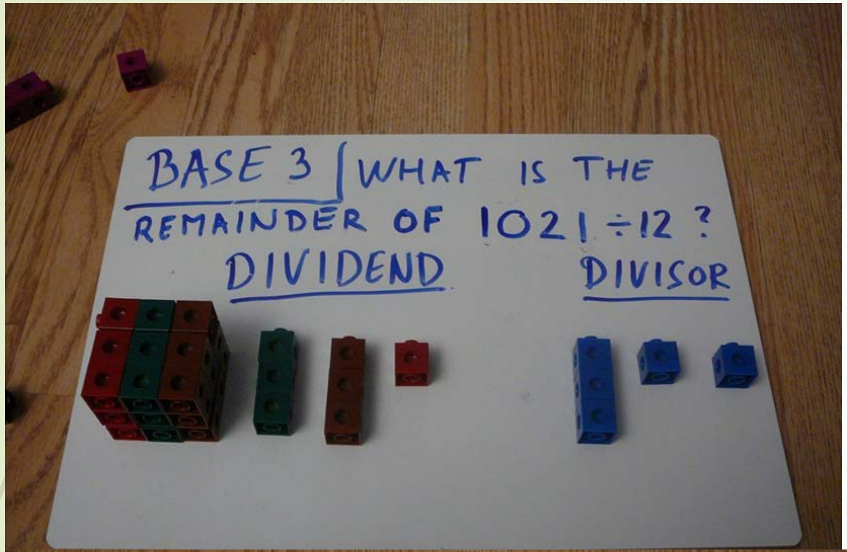




"Traditional" long division performed by volunteer student in front of the class

Generalization Task: Formulate a divisibility rule for divisibility by 2 in base *N*

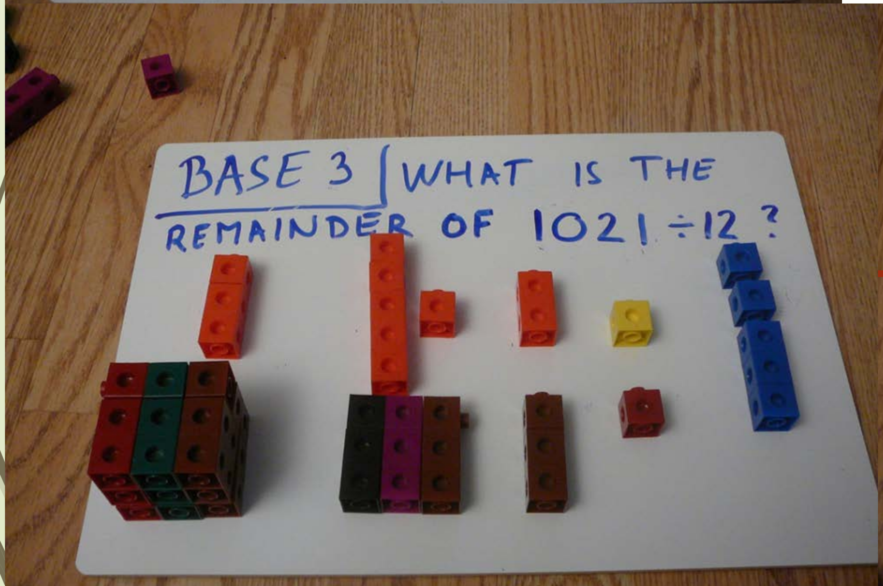
Resolution based on the behavior of the blocks

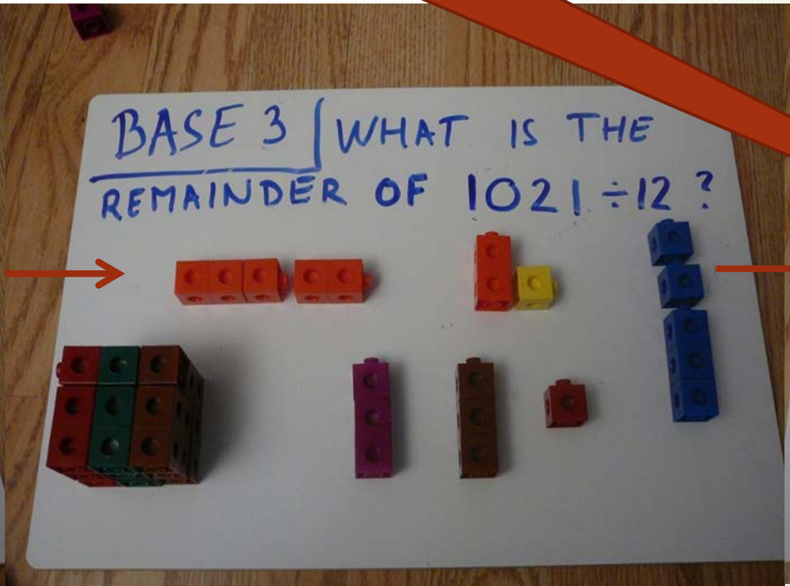


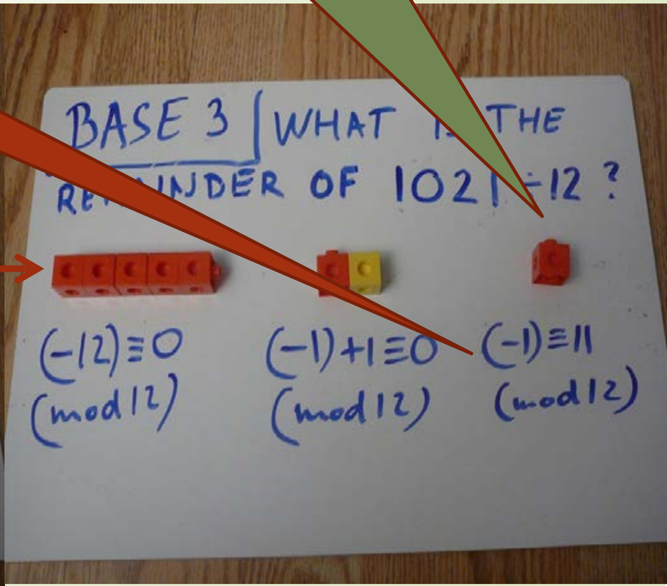
- The "**embodied way**" of consolidating the answers by **manipulatives** obtained by long division.
- Students are randomly called to critically evaluate the work of volunteers working at the board!

Formal Representation

How does one red/orange cube represent "11"?







Homework challenge

- **Task: devise a game** that promotes the understanding of certain divisibility properties (by their pupils) and describe the rules of the game.
- **Example**:): Take a deck of 52 cards plus 4 jokers to represent digits in base fifteen as follows: 2=2, … 10=10, J=eleven, Q=twelve, K=thirteen, A= fourteen OR one, Joker= any digit (including the digit 0). Each player receives 12 cards from a shuffled deck (with jokers added). One card is turned up from the deck to represent the requested divisor. Players take turns. When it is their turn, they pull a card from the deck or pick up the card the previous player discarded, and put down some or none of the cards in their hands as 1-, 2-, 3- , 4-, 5-, or 6-digit numbers that are divisible by the divisor and discard one card. The player who can first get rid of all her cards wins.
- **The game should be tried out** using at least 4 different players. The **strategies** and **reflections** of the players need to be **recorded**.
- The games needs to be possibly **improved** and **discussed** in class **based on** the way the players reacted in live **experiments**.

Synthetizing Varga and Dienes-1

- A synthesis that resolves the didactic dilemma at

the junction of

- **2. Teacher Training** — changing habitual practices
- **1. Classroom Teaching** — complex education reform

- Learning is internalized action that should be augmented by appropriate **games** and **manipulatives** at **both** levels.
  - New technological tools and the Internet of Things are to be considered as manipulatives.
  - **Games** are essential for both level-1 and level-2 **groupwork**
- Unbiased involvement in reflective mathematical and didactic practice develops **active** fellow-**learners**

Synthetizing Varga and Dienes-2

- Conducting demonstration classes remain an essential way of building consensus among researchers and practitioners on desired mathematical practices.
- Dienes's principles can be applied at **both** levels
  - in a similar vein to the overlapping material
  - the **deep-end-principle** applies to thematic embeddings

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